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A NOTE ON LEVI-MALCEV THEOREM FOR HOMOGENEOUS BOL ALGEBRAS

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Abstract

Bol algebras are a broad generalization of Lie algebras that include Lie, Malcev algebras, and Lie triple systems as very particular examples. We present the concept of solvable ideals, nilpotent ideals in the category of Bol algebras, the existence of radical and nilradical is proven. The Levi-Malcev theorem is established for homogeneous Bol algebras.

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1. Introduction

The class of Bol algebras was introduced by Sabinin and Mikhev [17] in connection with the study of tangent algebras of smooth Bol loops. It is well known that, Bol algebras generalize Malcev algebras, which are also natural generalization of Lie algebras. For the first time, the notion of Malcev algebra appeared in the remarkable work of Malcev was called Moufang-Lie algebra by Malcev himself, see [13]. The question of the validity of the analogue of Levi-Malcev theorem for Malcev algebras was posed by Kuz'min [9] in the Dnester note-books and the answer was given by Grishkov and Kuz'min affirmatively, see [4] and [10]. We had recall that, Bol algebras generalize Malcev algebras. Since the Levi-Malcev theorem plays the basic role in structural theories, it is useful to get some generalized version of such theorem for Bol algebras. However, the underline theory was incomplete because of the absence of a theorem analogous to that of Levi-Malcev in Malcev algebras. Till now, the validity of Levi-Malcev theorem is unknown for Bol algebras. That is the main investigation of this paper. First, we present the notion of solvability, nilpotency, and construct the radical and the nilradical for Bol algebras. Next, we establish the important results between the Killing-Ricci form, nilpotent Bol algebras, and automorphisms of Bol algebras. In the last section, we build Levi-Malcev theorem for homogeneous Bol algebras.

2. Fundamental Concepts

Definition 2.0.1. A vector space V over a field K equipped with a trilinear operation $[-; -, -]$ is called a Lie triple system, if

(i) $[a; a, b] = 0$,

(ii) $[a; b, c] + [b; c, a] + [c; a, b] = 0$,

(iii) $[x; y, [a, b, c]] = [[x; y, a]; b, c] + [a; [x; y, b], c] + [a, b, [x; y, c]]$ for all $x, y, z, a, b, c \in V$.

A left Bol algebras $(V, [-; -, -], [-, -])$ is a Lie triple system $(V, [-; -, -])$ with an additional skew-symmetric operation $[-, -]$ satisfying $[a, b, [x, y]] = [[a; b, x], y] + [x; [a; b, y]] + [x; y, [a, b]] + [[a, b], [x, y]]$ for all $x, y, z, a, b, c \in V$.

Example 2.0.1. (1) Let K be a commutative field of characteristic different to zero. We set $\mathcal{A} = \{f / f(x) = ax + b\}$, and we defined on $\mathcal{A}$, the following operations:

$[f, g, h] = f \frac{dg}{dx} \frac{dh}{dx} - g \frac{df}{dx} \frac{dh}{dx}$, and

$[f, g] = f \frac{dg}{dx} - g \frac{df}{dx}$,

$(\mathcal{A}, [-, -][-, -, -])$ is a Bol algebra.

(2) Let V be an algebra of dimension 3 with a given basis $(e_i)_{1 \leq i \leq 3}$, we defined by

$[e_1, e_3] = xe_1 + pe_2 + e_3$, for all x, p greater or equal to zero,

$(e_1, e_2, e_3) = e_1$,

$(e_1, e_3, e_3) = e_1$.

The bilinear and the trilinear brackets of those not appearing are set to be equal zero. $(V, [\ ,\ ], (-, -, -))$ is Bol algebra.

Definition 2.0.2. A homomorphism $f : V \to W$ between two Bol algebras V and W is a linear map preserving the ternary and the binary operations.

Definition 2.0.3. An ideal of Bol algebra V is a subspace I for which $[I, V] \subseteq I$ and $[I, V, V] \subseteq I$.

Remark 2.0.1. If I is an ideal of Bol algebra V, the quotient V / I is a Bol algebra with the operations: $[x + I; y + I, z + I] = [x; y, z] + I$ and

$[x + I, y + I] = [x, y] + I$ for all $x, y, z \in V$. The mapping $x \longmapsto x + I$ is a homomorphism of Bol algebra V onto a Bol algebra $\frac{V}{I}$ with the kernel I.

2.1. Solvable and semi-simple Bol algebras. As in the study of Lie triple systems and anti-Lie triple systems in [5, 7, 12], we introduce the notion of radical and nilradical for Bol algebras. The study of solvable and semi-simple Bol algebra can be find on [11].

If I is ideal of Bol algebras V, put $I^0 = I$, $I^{(1)} = [I, I] + [V, I, I]$, and $I^{(k+1)} = [I^{(k)}, I^{(k)}] + [V, I^{(k)}, I^{(k)}]$ for $k \in \mathbb{N}$.

Lemma 2.1.1 ([1]). *For each k in* $\mathbb{N}$, $I^{(k+1)}$ *is an ideal of* $I^{(k)}$. *Thus,* $I \supseteq I^{(1)} \supseteq \ldots \supseteq I^{(k+1)}$.

Definition 2.1.1. An ideal I in Bol algebra V is solvable, if there is a positive integer n for which $I^{(n)} = 0$.

It is well to remark that, if I and J are ideals in Bol algebra V, $J \subseteq I$ and I is solvable, then J also.

We note that, if φ is a homomorphism of Bol algebras V into a second Bol algebra W, then $\varphi(V^{(i)}) = (\varphi(V))^{(i)}$.

Lemma 2.1.2. *Every subalgebra and every homomorphic image of solvable Bol algebras is solvable. If V contains a solvable ideal I such that* V / I *is solvable, then V is solvable.*

Proof. The first two statements are clear. Let denotes by π the canonical projection $x \longmapsto x + I$ of V onto V / I. Then

$$\begin{aligned} \pi(V^{(k)}) &= (\pi(V))^{(k)} \\ &= (V / I)^{(k)} \\ &= 0, \end{aligned}$$

where k is the index of solvability of V / I. Hence $V^{(k)} \subseteq I$. According to the hypothesis, I is solvable and we have $I^{(h)} = 0$ for some h. Hence, $V^{(k)} \subseteq I$ implies V is solvable. □

Proposition 2.1.1. *The sum of any two solvable ideals of V is solvable ideal.*

Proof. Let I and J be solvable ideals. By one of the standard theorem of isomorphism theorems, $I \bigcap J$ is an ideal in I and $\frac{I+J}{I} \cong \frac{I}{I \bigcap J}$. The Bol algebra $\frac{I}{I \bigcap J}$ is solvable as homomorphic image of the solvable Bol algebra I. Since J is solvable, the lemma applies to prove that $I + J$ is solvable. □

Corollary 2.1.1. *Let us form* $R(V) = \sum_{\alpha \in \Lambda} I_\alpha$, I_α *is solvable ideal in V and* Λ *is a subset of* $\mathbb{N}$. *Then,* $R(V)$ *is solvable and is the unique maximal solvable ideal in V.*

Proof. We are going to proceed by induction.

If the cardinal of Λ is 2. According to Proposition 2.1.1, $R(V)$ is solvable. Assume that the sum $\sum_{\alpha \in \Lambda_1} I_\alpha$ is solvable with Λ_1 a subset of Λ, which has k elements.

Let Λ_2 be a subset of Λ, which has $(k+1)$ elements and $\Lambda_1 \subseteq \Lambda_2$. We have $\sum_{\alpha \in \Lambda_2} I_\alpha = \sum_{\alpha \in \Lambda_1} I_\alpha + \alpha_{k+1}$, which is solvable as sum of two solvable ideal.

Let L be a solvable ideal of V, then there is $\alpha \in \Lambda$ such that $L = I_\alpha$. Hence $R(V)$ is maximal ideal in V.

□

Definition 2.1.2. The ideal $R(V)$ established above is called the radical of Bol algebras V. In case $R(V) = 0$, the Bol algebra V is semi-simple.

2.2. Nilpotent ideal in Bol algebras. In this subsection, we extend Hopkins definition for the Lie triple systems [5], to Bol algebras. There is an other definition of nilpotence for Lie triple systems given by Kamiya, see [7]. Suppose that L is an ideal of V. Then $L^{[0]} = L$ and for $k \geq 0$, $L^{[k+1]} = [V, L^{[k]}] + [L^{[k]}, V, L] + [L, V, L^{[k]}]$ defines the series for L. L is nilpotent if $L^{[n]} = 0$ for some n. Note that $[L, L^{[k]}, V] + [V, L^{[k]}, L] \subseteq L^{[k+1]}$.

Lemma 2.2.1. *I and J are two ideals of Bol algebra V such that $I \subseteq J$. If J is nilpotent, then I also is nilpotent.*

Proof. We show by induction that, $I^{[k]} \subseteq J^{[k]}$. $I^{[0]} = I$ and $J^{[0]} = J$, the property is true for $k = 0$.

Suppose that $I^{[k]} \subseteq J^{[k]}$ and let us show that, $I^{[k+1]} \subseteq J^{[k+1]}$. We have $I^{[k+1]} = [V, I^{[k]}] + [I^{[k]}, V, I] + [I, V, I^{[k]}]$ and $J^{[k+1]} = [V, J^{[k]}] + [J^{[k]}, V, J] + [J, V, J^{[k]}]$. Since $I^{[k]} \subseteq J^{[k]}$, then $I^{[k+1]} \subseteq J^{[k+1]}$.

□

Proposition 2.2.1. *Suppose that L and J are ideals of Bol algebra V, the following statements are true:*

(1) $L^{[k]}$ *is an ideal of V, so* $L^{[k+1]} \subseteq L^{[k]}$.

(2) $(L \cap V)^{[k]} = L^{[k]} \cap V$.

(3) *If L and J are nilpotent, then* $L + J$ *is a nilpotent ideal of V.*

Proof. (1) is proven as Lemma 2.1.1.

(2) We proceed by induction. If $k = 0$, we have $(L \cap V)^{[0]} = L^{[0]} \cap V$. Suppose that $(L \cap V)^{[k]} = L^{[k]} \cap V$. We have

$$
\begin{aligned}
L^{[k+1]} &= ([V, L^{[k]}] + [L^{[k]}, V, L] + [L, V, L^{[k]}]) \cap V \\
&= [V, L^{[k]} \cap V] + [L^{[k]} \cap V, V, L \cap V] + [L \cap V, V, L^{[k]} \cap V] \\
&= [V, (L \cap V)^{[k]}] + [(L \cap V)^{[k]}, V, L \cap V] + [L \cap V, V, (L \cap V)^{[k]}] \\
&= (L \cap V)^{[k+1]}.
\end{aligned}
$$

Hence the result.

(3) We can show by induction, the relation of inclusion

$$(L + J)^{[k]} = (L)^{[k]} + (J)^{[k]} + \sum_{i=1}^{k} L^i \cap J^i. \qquad \square$$

Let V be a finite dimensional Bol algebra, then this implies that there is in V a unique maximal nilpotent ideal, called the nilradical of V, which contains all other nilpotent ideal of V.

We note that, if φ is a homomorphism of Bol algebras V into a second Bol algebra W, then $\varphi(V^{[i]}) = (\varphi(V))^{[i]}$.

We set now, $Z_0(V) = \{x \in V/[a, x] = 0, \forall a \in V\}$ and $Z_1(V) = \{x \in V/[a, b, x] = 0, \forall a, b \in V\}$. $Z_0(V) \cap Z_1(V)$ is an ideal of Bol algebra V and we denotes $Z = Z_0(V) \cap Z_1(V)$.

Proposition 2.2.2. *If V/Z is nilpotent, then V is nilpotent.*

Proof. Suppose that V/Z is nilpotent of nilindex n, and $\pi : V \longrightarrow V/Z$ the canonical projection. Then $V^{[n]} \subseteq Z$.

$\square$

If I is an ideal of V, we can define another weak series for I as above. We set $I^0 = I$ and $I^{k+1} = [I^k, I^k]$ for $n \geq 0$. We have also $\varphi(V^i) = (\varphi(V))^i$, if φ is a morphism of Bol algebra.

3. Bol Algebras with Killing-Ricci Form

The Killing-Ricci form was originally introduced by Cartan into a Lie algebra theory in his thesis [2], to characterize solvable and semi-simple Lie algebras. Following Cartan and Kikkawa [8] for the case of Lie triple systems, Killing-Ricci form was introduced by Kuz'min and Zaidi on Bol algebras, for the same purpose, see [11], but they are not characterized the automorphisms of Bol algebras and the nilpotent Bol algebras with such form; that is the aim of this section.

Let $(B, \cdot, [-, -, -])$ be a Bol algebra. A linear map $p : B \longrightarrow B$ is called pseudoderivation of B, with the companion a, if the following equalities hold for all $x, y, z \in B$:

$$P([x, y, z]) = [p(x), y, z] + [x, p(y), z] + [x, y, p(z)],$$

$$p(x \cdot y) = p(x) \cdot y + x \cdot p(y) + [x, y, a] + a \cdot (x \cdot y).$$

For all $a, b \in B$, the map $D(a, b) : x \longmapsto [a, b, x]$ is a pseudoderivation with companion $a \cdot b$; linear combinations of such maps are called inner pseudoderivation and denoted by $IPderB$. The standard enveloping Lie algebra for B is the Lie algebra $L = B + [B, B] = B + IPderB$, with the bracket defined by

$$[a, b] = a \cdot b + D(a, b), [D(a, b), c] = D(a, b)(c), \text{ and}$$

$$[D(a, b), D(c, d)] = D(a, b)D(c, d) - D(c, d)D(a, b).$$

This enveloping Lie algebra is the same for the associate Lie triple system and was defined by Hopkins in [5]. It is important to remark that Hopkins has defined the standard enveloping Lie algebra in general case of Lie and anti-Lie triple systems; for Lie triple systems, the standard enveloping is Lie algebra, and for anti-Lie triple systems, the standard enveloping is Lie superalgebra. To know more about Lie superalgebras, see [6].

Now, let B be a finite-dimensional Bol algebra, L be its standard enveloping Lie algebra, and α be the Killing-Ricci form of L. We define the Killing-Ricci form of the Bol algebra B as the restriction β of the form α to B. The following theorem provide the link between automorphisms of Bol algebras and the Killing-Ricci form for such Bol algebra as in the case of Lie theory. Our first main result is this.

Theorem 3.0.1. *Let B be a finite-dimensional Bol algebra. The Killing-Ricci form on Bol algebra B is invariant under the automorphisms of Bol algebras; that is, $\beta(f(x), f(y)) = (x, y)$ for all $x, y \in B$ and f an automorphism of B.*

Proof. Let $f : B \longrightarrow B$ be an automorphism of Bol algebra B and $L = B + [B, B]$ its standard enveloping Lie algebra. We define

$$\widetilde{f} : L \longrightarrow L$$

as follow:

$\widetilde{f}(x) = f(x)$, for $x \in B$; $\widetilde{f}(D(a, b)) = D(f(a), f(b))$ and $\widetilde{f}(D(a, b)D(c, d))$ $= D(f(a), f(b))D(f(c), f(d))$, we extend $\widetilde{f}$ to all elements by linearity.

Let a, b, c, d on B, we have

$$\begin{aligned}\widetilde{f}([a, b]) &= \widetilde{f}(a \cdot b + D(a, b)) \\ &= \widetilde{f}(a \cdot b) + \widetilde{f}(D(a, b)) \\ &= f(a) \cdot f(b) + D(f(a), f(b)) \\ &= [\widetilde{f}(a), \widetilde{f}(b)],\end{aligned}$$

$$\begin{aligned}\tilde{f}([D(a, b), c]) &= \tilde{f}(D(a, b)(c)) \\ &= f([a, b, c]) \\ &= [f(a), f(b), f(c)] \\ &= D(f(a), f(b))(f(c)) \\ &= [D(f(a), f(b)), f(c)] \\ &= [\tilde{f}(D(a, b)), \tilde{f}(c)],\end{aligned}$$

$$\begin{aligned}\tilde{f}([D(a, b), D(c, d)]) &= \tilde{f}(D(a, b)D(c, d)) - \tilde{f}(D(c, d)D(a, b)) \\ &= D(f(a), f(b))D(f(c), f(d)) - D(f(c), f(d))D(f(a), f(b)) \\ &= [D(f(a), f(b)), D(f(c), f(d))] \\ &= [\tilde{f}(D(a, b)), \tilde{f}(D(c, d))].\end{aligned}$$

Let α be the Killing-Ricci form of L and the Killing-Ricci form of the Bol algebra B is the restriction β of the form α to B. $\tilde{f}$ is an automorphism of L. Then $\alpha(\tilde{f}(x), \tilde{f}(y)) = (x, y)$. Hence $\beta(f(x), f(y)) = (x, y)$.

□

Now we can also give the relation between nilpotent Bol algebra and its Killing-Ricci form.

Lemma 3.0.1. *Let B be a Bol algebra and L be an enveloping Lie algebra for B, $\mathcal{N}(B)$ and $\mathcal{N}(L)$, respectively, the nilradical of B and L. If I is a nilpotent ideal of B, then $I + [B, I]$ is a nilpotent ideal of L; in particular, $\mathcal{N}(B) \subseteq \mathcal{N}(L)$.*

Proof. Since any nilpotent ideal of a Bol algebra is nilpotent ideal in its restriction Lie triple systems. This lemma follows from a similar statement about Lie triple systems, see ([5], Corollary 3.3). □

Theorem 3.0.2. *The Killing-Ricci form of a nilpotent Bol algebra B is identically zero.*

Proof. Let B be a nilpotent Bol algebra and L be the enveloping Lie algebra for B. Let β and α be the Killing-Ricci form, respectively, of B and L. B is nilpotent, so is L, according to Lemma 3.0.1. It is clear that $\alpha = 0$, hence $\beta = 0$.

□

4. The Levi-Malcev Theorem for Homogeneous Bol Algebras

The definition of homogeneous Bol algebras in this paper is the one of that was used by Filippov in [3].

Definition 4.0.1. Bol algebras, whose trilinear operation is expressible through the bilinear operation as homogeneous polynomial are called homogeneous Bol algebras; that is,

$$[x, y, z] = \alpha[[x, y], z] + \beta[[y, z], x] + \gamma[[z, x], y],$$

for some α, β, γ in the based field and $x, y, z \in V$.

We have also a geometric realization of homogeneous Bol algebras, given by Sabinin in [19] and Mikhev in [14]. Let (B, ∇) be an affinely connected space with the curvature tensor $R = 0$, and let $X(x)$ be an infinitesimal affine transformation in (B, ∇); that means for every vector field $Y \in B$, we have $L_X \circ \nabla - \nabla \circ L_X = \nabla_{[X, Y]}$, where L is a Lie differential in the direction of the vector field X. Let us introduce in B a basis of vectors field such that for every $X_1, X_2, \ldots, X_n$, then $X(x) = \alpha^j(x)(X_j)|_x$ and we can have $[X_i, X_j] = -T^j_{ik}X_j$. The following definition is due to Mikhev in [14].

Definition 4.0.2. A local analytic loop $(B, ., e)$ verify the G-property if there exists such a neighbourhood U of the point $e \in B$, such that for any $x \in U$, the loops $(B, ., e)$ and $(B, \frac{1}{x}, e)$ are isomorphic.

The underline author has show that, a local analytic Bol loop $(B, ., e)$ posses a G-property, if and only if the corresponding affine connected space (B, ∇) is local homogeneous. To know more about homogeneous spaces, see [15]. Now, let B be a Bol loop that posses a G-property and $V = T_e(B)$ be its an infinitesimal object. Indeed, we will have the following bilinears and trilinears operations:

$$(\zeta, \eta) \longmapsto \zeta \cdot \eta \in T_e(B), \quad (\zeta \cdot \eta)^i = T^j_{ik}(e)\zeta^j\eta^k,$$

$$(\zeta, \eta, \tau)^i = (\nabla_e T^j_{ik} + T^s_{ik}T^i_{sl})\zeta^j\eta^k\tau^l,$$

$[X_i, X_j] = -T^j_{ik}X_j,$ which will verify the identities of the definition of Bol algebras. In addition, this Bol algebras is homogeneous.

4.1. Connection with Malcev algebras. The notions of ideal, solvable ideals, radical for Malcev algebras was defined in [20]. In this subsection, we show that, there is compatibility for the notion of ideal, solvability, radical between homogeneous Bol algebras and Malcev algebras.

An anticommutative algebra $(M, [,])$ is said to be a Malcev algebra, if it satisfies the identity $[J(x, y, z), x] = J(x, y, [x, z]),$ where

$$J(x, y, z) = [x, [y, z]] + [y, [z, x]] + [z, [x, y]]$$

is the Jacobian of $x, y, z,$ see [4, 13, 16, 18, 19]. Malcev algebras arise as tangent algebras of analytic Moufang loop, it is a geometric realization of Malcev algebras, see the same references above.

From Malcev algebra $(M, [,]),$ one can obtain a Bol algebra $(M, [,], [, ,])$ with the ternary product defined by

$$[x,\ y,\ z] = [[x,\ y],\ z] - \frac{1}{3} J(x,\ y,\ z).$$

In fact, this Bol algebra is homogeneous.

We recall that, Malcev algebras are particular examples of homogeneous Bol algebras; thus, every homogeneous Bol algebra is neither Malcev algebra, or not.

We have the following theorem of Filippov in [3], which give a nice characterization of the class of homogeneous Bol algebras, which are not Malcev.

Theorem 4.1.1 (Filippov [3])**.** *If the homogeneous Bol algebra V over a field of characteristic distinct from* 2 *and* 3 *is not Malcev algebra*, *then* $(V^2)^3 = 0$.

Now we are in position to prove our second main result.

Theorem 4.1.2. *Suppose that V is a finite-dimensional homogeneous Bol algebra over a field of characteristic zero and* $R(V)$ *is the solvable radical of V*, *then V decompose into a semi-direct sum of* $R(V)$ *and some semi-simple Bol sub-algebra Q of V.*

The proof of this theorem use the following lemmas:

Lemma 4.1.1. *Let V be a homogeneous Bol algebra which is Malcev algebra*, *then the following statements are equivalent*:

(1) *I is an ideal of Malcev algebra V.*

(2) *I is an ideal of Bol algebra V.*

Proof. Suppose that, I is an ideal of Malcev algebra V; then $[I,\ V] \subseteq I$. We have

$$\begin{aligned}[I,\ V] + [I,\ V,\ V] &= [I,\ V] + \alpha[[I,\ V],\ V] + \beta[[V,\ V],\ I] + \gamma[[V,\ I],\ V] \\ &= [I,\ V] + [[I,\ V],\ V] + [[V,\ V],\ I] + [[V,\ I],\ V] \\ &\subseteq I.\end{aligned}$$

Conversely, let I be an ideal of Bol algebra V, then $[I, V] + [I, V, V] \subseteq I$; thus $[I, V] \subseteq I$ and I is an ideal of Malcev algebra.

□

Lemma 4.1.2. *Let V be a homogeneous Bol algebra, which is Malcev algebra, then the following statements are equivalent*:

(1) *I is a solvable ideal in Malcev algebra V of index m.*

(2) *I is a solvable ideal of Bol algebra V of index m.*

Proof. Let us denoted by I^k the ascendant series of ideals I in Malcev algebra v by: $I^0 = I$ and $I^{k+1} = [I^k, I^k]$ for $n \geq 0$, and recall the ascendant series of ideals I in Bol algebra V.

Let us show by induction that $I^k = 0$, if and only if $I^{[k]} = 0$, $\forall n \geq 1$.

If $n = 1$, we have

$$\begin{aligned} I^1 &= [I, I] + [V, I, I] \\ &= [I, I] + \alpha[[V, I], I] + \beta[[I, I], V] + \gamma[[I, V], I] \\ &= [I, I] + [[V, I], I] + [[I, I], V] + [[I, V], I] \\ &= 0. \end{aligned}$$

Then $I^1 = 0$ implies $I^{(1)} = 0$.

Conversely, assume that $I^{(1)} = 0$, i.e., $I^1 + [V, I, I] = 0$. Then $I^1 = 0$. Let I be an ideal of index m in Malcev algebra. We have

$$\begin{aligned} I^{[m]} &= I^m + P([V, I^m]) \\ &= 0, \end{aligned}$$

where P is homogeneous polynomial.

Conversely, if $I^{(m)} = 0$, then $I^m + P([V, I^m]) = 0$, thus $I^m = 0$. □

Let V be a Bol algebra, we denote by $R(V)^B$ it's radical. And let V be a Malcev algebra, we denote by $R(V)^M$ it's radical. We have the following lemma:

Lemma 4.1.3. *If V is a Bol algebra which is also Malcev algebra, then* $R(V)^M = R(V)^B$.

Proof. We have show that if I is a subset of V, I is an ideal of Bol algebra V, if and only if I is an ideal of Malcev algebra v; and I is solvable on Bol algebra V of index n, if and only if I is solvable on Malcev algebra V of index n. Hence $R(V)^M = R(V)^B$.

□

Corollary 4.1.1. *Let V be a Bol algebra, which is Malcev algebra. An ideal I of Malcev algebra V is semi-simple, if and only if I is a semi-simple ideal of Bol algebra V.*

Proof. We use the above theorem in the case, which

$$R(V)^M = R(V)^B = 0.$$

□

Lemma 4.1.4. *Every ideal in the finite dimensional homogeneous Bol algebra, which is not Malcev is solvable.*

Proof. Let V be a homogeneous Bol algebra, which is not Malcev algebra. We have $V^0 = V$

$$\begin{aligned}
V^{(1)} &= [V, V] + [V, V, V] \\
&= [V, V] + \alpha[[V, V], V] + \beta[[V, V], V] + \gamma[[V, V], V] \\
&= [V, V] + [[V, V], V] \\
&= [V, V] \\
&= V^1.
\end{aligned}$$

$$
\begin{aligned}
V^{(2)} &= [[V, V], [V, V]] + [V, [V, V], [V, V]] \\
&= [[V, V], [V, V]] + [V, V] + \alpha[[V, [V, V]], [V, V]] \\
&\quad + \beta[[V, V], [V, V], V] + \gamma[[[V, V], V], [V, V]] \\
&= [[V, V], [V, V]] + [[V, [V, V]], [V, V]] \\
&= [[V, V], [V, V]] \\
&= V^2. \\
V^{(3)} &= [V^{(2)}, V^{(2)}] + [V, V^{(2)}, V^{(2)}] \\
&= [[[V, V], [V, V]], [[V, V], [V, V]]] + \alpha[[V, V^{(2)}], V^{(2)}] \\
&\quad + \beta[[V^{(2)}, V^{(2)}], V] + \gamma[[V^{(2)}, V], V^{(2)}] \\
&= [[[V, V], [V, V]], [[V, V], [V, V]]] \\
&\quad + [[V, [[V, V], [V, V]], [[V, V], [V, V]]] \\
&\quad + [[[[V, V], [V, V]], [[V, V], [V, V]]], V] \\
&\quad + [[[[V, V], [V, V]], V], [[V, V], [V, V]]] \\
&= V^8 + V^6 + V^9 + V^6 \\
&= 0.
\end{aligned}
$$

According to Filippov [3].

Let I be an ideal on algebra V, $I \subseteq V$, and V is solvable. Hence I is solvable. □

Now we are going to prove our theorem.

Proof. Let V be a homogeneous Bol algebra. If V is Malcev algebra, according to the theorem of Griskov-Kuz'min in [4] and [10], $V = R(V) \oplus Q$, where $R(V)$ is a radical of Bol algebra V and Q is a semi-simple sub-Bol algebra of V. Since the Lemmas 4.1.1-4.1.3 show the compatibilities of the two structures.

If V is not Malcev algebra, according to the Lemma 4.1.4, V is solvable. Hence the theorem is valid.

□

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